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SUMMARY

In this article, a semi-analytical theory of general planetary perturbations, which is somewhat akin to Hill's theory, is developed. In both methods the first order perturbations coincide, but the theories of perturbations of higher orders are different. The inconveniences of Hill's method, namely, the triple integral in the perturbations of the radius vector and the redundant constant of integration, do not appear here. The short and the long period terms containing the squares of the small divisors are localized and combined together. The existence of such a direct way of separating these important terms from the remaining perturbations constitutes a significant characteristic of a planetary theory. The form of decomposition of perturbations as used in this article leads to a system of differential equations easily integrable by Hill's procedure and to a symmetrical scheme for the computation of perturbations of higher orders.

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INTRODUCTION

In this article a numerical theory of general planetary perturbations is developed. The perturbations are obtained in the standard form of series containing the periodic, the secular, and the mixed terms. The coefficients of terms are numerical; they are obtained by double harmonic analysis as applied to the force components, and subsequent integration. The numerical theory of perturbations in the coordinates escapes the inconveniences of the numerical theory in the elements; for example, since the eccentricity and the sine of the inclination do not appear as divisors in the differential equations, no numerical difficulties arise in the case of nearly circular orbits or lowly inclined orbits.

The method presented here is somewhat akin to Hill's method (Reference 1), at least where the first order perturbations are concerned; in both methods the first order perturbations coincide. The theories of perturbations of higher orders in both methods are different.

We determine the perturbations in rectangular coordinates directly without using the perturbations in polar coordinates as an intermediary means, as is done in Hill's method. Some other inconveniences are removed in the theory presented here: The triple integral and the seventh constant of integration so peculiar to Hill's method do not appear; the components of perturbations along the radius vector are determined in a more direct manner; and the difficulties associated with determining the redundant constant of integration vanish.

In Hill's method the undisturbed true anomaly is taken as the independent variable. Such a choice causes numerous inconveniences if the perturbations in the motion of the disturbing body are also to be taken into account. For this reason the universal variable, time, is used in our exposition.

Since the advent of electronic machines, general perturbation theories can successfully compete with numerical integration procedures. The theory of Mars in Hansen's coordinates recently developed by Clemence (References 2 and 3) brilliantly confirms this statement. The reference ellipse representing the undisturbed motion can be chosen in a variety of ways. The only restriction imposed is that the difference between the disturbed and undisturbed motions be small, of the order of the perturbations.

In the article by Musen and Carpenter (Reference 4) a decomposition of the perturbations in the position vector along $\vec{r}$, $\vec{v}$, and $\vec{R}$ was suggested. In the present work we suggest a decomposition along $\vec{r}^{\circ}$, $\vec{R}\times\vec{r}^{\circ}$, and $\vec{R}$. This form of decomposition leads to a more compact and more symmetrical scheme than the author's previous scheme for developing perturbations of higher order.

BASIC DIFFERENTIAL EQUATIONS

The equation of the motion of the planet m as referred to the rotating undisturbed frame $\vec{r}^{\circ}$, $\vec{R}\times\vec{r}^{\circ}$, $\vec{R}$ has the form*

$$\frac{d^2\,\delta\vec{r}}{dt^2} + 2\vec{\psi}\times\frac{d\delta\vec{r}}{dt} + \vec{\psi}\times\left(\vec{\psi}\times\delta\vec{r}\right) + \frac{d\vec{\psi}}{dt}\times\delta\vec{r}$$

$$= \mu^2\,\nabla\left(\frac{1}{|\vec{r}+\delta\vec{r}|} - \frac{1}{r}\right) + \mu^2\,\nabla\Omega\left(\vec{r}+\delta\vec{r},\ \vec{r}\,'+\delta\vec{r}\,'\right)\ . \tag{1}$$

Substituting

$$\vec{\psi} = \frac{\mu\sqrt{p}}{r^2}\,\vec{R}\ ,$$

$$\frac{d\vec{\psi}}{dt} = -\frac{2\mu\sqrt{p}}{r^3}\frac{dr}{dt}\,\vec{R}$$

into Equation 1, introducing the differential operator

$$D = \delta\vec{r}\cdot\nabla + \delta\vec{r}\,'\cdot\nabla'\ ,$$

and taking the equation

$$\nabla D\,\frac{1}{r} = -\frac{\delta\vec{r}}{r^3} + \frac{3\vec{r}\,\vec{r}\cdot\delta\vec{r}}{r^5}$$

into account, we can rewrite Equation 1 in the form

$$\frac{d^2\,\delta\vec{r}}{dt^2} + \frac{2\mu\sqrt{p}}{r^2}\,\vec{R}\times\frac{d\delta\vec{r}}{dt} + \frac{\mu^2 p}{r^4}\,\vec{R}\times\left(\vec{R}\times\delta\vec{r}\right)$$

$$-\frac{2\mu\sqrt{p}}{r^3}\frac{dr}{dt}\,\vec{R}\times\delta\vec{r} + \mu^2\left(\frac{\delta\vec{r}}{r^3} - \frac{3\vec{r}\,\vec{r}\cdot\delta\vec{r}}{r^5}\right) = \mu^2\,\vec{F}\ , \tag{2}$$

where

$$\vec{F} = \nabla\left(E^D - 1 - D\right)\frac{1}{r} + \nabla E^D\,\Omega\left(\vec{r},\ \vec{r}\,'\right)\ . \tag{3}$$

*Notations are defined in Appendix A.

In applying the operators ∇ or ∇', we consider every function as a function of $\vec{r}$ and $\vec{r}\,'$ only; and the perturbations $\delta\vec{r}$ and $\delta\vec{r}\,'$ are considered as relative constants. The operator E^D accomplishes the development of Equation 3 into a power series with respect to the perturbations.

Decomposing $\delta\vec{r}$ along the axes $\vec{r}^{\circ}$, $\vec{R}\times\vec{r}^{\circ}$, $\vec{R}$ and putting

$$\delta\vec{r} = \xi\,\vec{r}^{\circ} + \eta\,\vec{R}\times\vec{r}^{\circ} + \zeta\,\vec{R}\,, \tag{4}$$

we have in the relative motion

$$\frac{d\delta\vec{r}}{dt} = \frac{d\xi}{dt}\,\vec{r}^{\circ} + \frac{d\eta}{dt}\,\vec{R}\times\vec{r}^{\circ} + \frac{d\zeta}{dt}\,\vec{R}\,, \tag{5}$$

$$\frac{d^2\,\delta\vec{r}}{dt^2} = \frac{d^2\xi}{dt^2}\,\vec{r}^{\circ} + \frac{d^2\eta}{dt^2}\,\vec{R}\times\vec{r}^{\circ} + \frac{d^2\zeta}{dt^2}\,\vec{R}\,. \tag{6}$$

Substituting Equations 4 through 6 into 2, we obtain

$$\frac{d^2\xi}{dt^2} - \frac{2\mu\sqrt{p}}{r^2}\frac{d\eta}{dt} - \mu^2\left(\frac{p}{r^4} + \frac{2}{r^3}\right)\xi + \frac{2\mu\sqrt{p}}{r^3}\frac{dr}{dt}\,\eta = \mu^2\,\Xi\,, \tag{7}$$

$$\frac{d^2\eta}{dt^2} + \frac{2\mu\sqrt{p}}{r^2}\frac{d\xi}{dt} - \frac{2\mu\sqrt{p}}{r^3}\frac{dr}{dt}\,\xi - \mu^2\left(\frac{p}{r^4} - \frac{1}{r^3}\right)\eta = \mu^2\,H\,, \tag{8}$$

$$\frac{d^2\zeta}{dt^2} + \frac{\mu^2}{r^3}\,\zeta = \mu^2\,Z\,, \tag{9}$$

where we put

$$\Xi = \vec{r}^{\circ}\cdot\vec{F}\,,$$

$$H = \vec{R}\times\vec{r}^{\circ}\cdot\vec{F}\,,$$

$$Z = \vec{R}\cdot\vec{F}\,.$$

We shall make use of the standard method of developing the perturbations into power series with respect to the disturbing mass and put

$$\delta\vec{r} = \vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \cdots\,,$$

$$\vec{r}_K = \xi_K\,\vec{r}^{\circ} + \eta_K\,\vec{R}\times\vec{r}^{\circ} + \zeta_K\,\vec{R}\,,$$

$$\xi = \xi_1 + \xi_2 + \xi_3 + \cdots\,,$$

$$\eta = \eta_1 + \eta_2 + \eta_3 + \cdots ,$$

$$\zeta = \zeta_1 + \zeta_2 + \zeta_3 + \cdots , \tag{10}$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots , \tag{11}$$

where $\vec{r}_\kappa$, ξ_κ, η_κ, ζ_κ, $\vec{F}_\kappa$ are of the order κ in m'.

In the previous article by Musen and Carpenter (Reference 4), on the basis of the formula

$$\begin{aligned}
\nabla E^D = \nabla &+ \left[\vec{r}_1 \cdot \nabla\nabla + \vec{r}_1{}' \cdot \nabla' \nabla\right] \\
&+ \left[\left(\vec{r}_2 \cdot \nabla\nabla + \vec{r}_2{}' \cdot \nabla' \nabla\right) + \frac{1}{2}\left(\vec{r}_1 \cdot \nabla + \vec{r}_1{}' \cdot \nabla'\right)^2 \nabla\right] \\
&+ \left[\left(\vec{r}_3 \cdot \nabla\nabla + \vec{r}_3{}' \cdot \nabla' \nabla\right) + \left(\vec{r}_1 \cdot \nabla + \vec{r}_1{}' \cdot \nabla'\right)\left(\vec{r}_2 \cdot \nabla + \vec{r}_2{}' \cdot \nabla'\right)\nabla \right. \\
&\left. + \frac{1}{6}\left(\vec{r}_1 \cdot \nabla + \vec{r}_1{}' \cdot \nabla'\right)^3 \nabla\right] + \cdots ,
\end{aligned}$$

it was found that

$$\vec{F}_1 = \nabla\Omega = \frac{m'}{1+m}\left(\frac{\vec{\rho}}{\rho^3} - \frac{\vec{r}'}{r'^3}\right) , \tag{12}$$

$$\begin{aligned}
\vec{F}_2 = {} & \frac{3\vec{r} \cdot \vec{r}_1 \vec{r}_1}{r^5} + \frac{3}{2}\frac{\vec{r}\,\vec{r}_1 \cdot \vec{r}_1}{r^5} - \frac{15}{2}\frac{\vec{r}\left(\vec{r} \cdot \vec{r}_1\right)^2}{r^7} \\
& + \vec{r}_1 \cdot \nabla\nabla\Omega + \vec{r}_1{}' \cdot \nabla' \nabla\Omega ,
\end{aligned} \tag{13}$$

$$\begin{aligned}
\vec{F}_3 = {} & \frac{3}{r^5}\left(\vec{r} \cdot \vec{r}_1 \vec{r}_2 + \vec{r}\,\vec{r}_1 \cdot \vec{r}_2 + \vec{r} \cdot \vec{r}_2 \vec{r}_1\right) \\
& - \frac{15}{r^7}\vec{r}\,\vec{r}_1 \cdot \vec{r}\,\vec{r}_2 \cdot \vec{r} + \frac{3}{2}\frac{\vec{r}_1 \vec{r}_1 \cdot \vec{r}_1}{r^5} \\
& - \frac{15}{2}\frac{\vec{r}\,\vec{r} \cdot \vec{r}_1 \vec{r}_1 \cdot \vec{r}_1}{r^7} - \frac{15}{2}\frac{\vec{r}_1\left(\vec{r} \cdot \vec{r}_1\right)^2}{r^7} \\
& + \frac{35}{2}\frac{\vec{r}\left(\vec{r} \cdot \vec{r}_1\right)^3}{r^9} \\
& + \left[\left(\vec{r}_2 \cdot \nabla\nabla\Omega + \vec{r}_2{}' \cdot \nabla' \nabla\Omega\right) + \frac{1}{2}\left(\vec{r}_1 \cdot \nabla + \vec{r}_1{}' \cdot \nabla'\right)^2 \nabla\Omega\right] .
\end{aligned} \tag{14}$$

By substituting

$$\vec{r}_1 = \xi_1 \vec{r}^\circ + \eta_1 \vec{R} \times \vec{r}^\circ + \zeta_1 \vec{R}$$

and

$$\nabla\nabla\Omega = \nabla \frac{m'}{1+m}\left(\frac{\vec{\rho}}{\rho^3} - \frac{\vec{r}'}{r'^3}\right) = \frac{m'}{1+m}\left(-\frac{I}{\rho^3} + \frac{3\vec{\rho}\vec{\rho}}{\rho^5}\right),$$

$$\nabla'\nabla\Omega = \nabla' \frac{m'}{1+m}\left(\frac{\vec{\rho}}{\rho^3} - \frac{\vec{r}'}{r'^3}\right) = \frac{m'}{1+m}\left(\frac{I}{\rho^3} - \frac{3\vec{\rho}\vec{\rho}}{\rho^5}\right) - \frac{m'}{1+m}\left(\frac{I}{r'^3} - \frac{3\vec{r}'\vec{r}'}{r'^5}\right)$$

into Equation 13, we deduce a compact expression for $\vec{F}_2$:

$$\vec{F}_2 = \frac{1}{r^4}\left[\left(-3\xi_1^2 + \frac{3}{2}\eta_1^2 + \frac{3}{2}\zeta_1^2\right)\vec{r}^\circ + 3\eta_1\xi_1\vec{R}\times\vec{r}^\circ + 3\zeta_1\xi_1\vec{R}\right]$$

$$+ \frac{m'}{1+m}\left[\left(\frac{\vec{\rho}_1}{\rho^3} - \frac{3\vec{\rho}\vec{\rho}\cdot\vec{\rho}_1}{\rho^5}\right) - \left(\frac{\vec{r}_1'}{r'^3} - \frac{3\vec{r}'\vec{r}'\cdot\vec{r}_1'}{r'^5}\right)\right].$$

Substituting Equations 10 and 11 into Equations 7 through 9, we obtain the equations

$$\frac{d^2\xi_\kappa}{dt} - \frac{2\mu\sqrt{p}}{r^2}\frac{d\eta_\kappa}{dt} - \mu^2\left(\frac{p}{r^4} + \frac{2}{r^3}\right)\xi_\kappa + \frac{2\mu\sqrt{p}}{r^3}\frac{dr}{dt}\eta_\kappa = \mu^2\Xi_\kappa,$$

$$\frac{d^2\eta_\kappa}{dt^2} + \frac{2\mu\sqrt{p}}{r^2}\frac{d\xi_\kappa}{dt} - \frac{2\mu\sqrt{p}}{r^3}\frac{dr}{dt}\xi_\kappa - \mu^2\left(\frac{p}{r^4} - \frac{1}{r^3}\right)\eta_\kappa = \mu^2 H_\kappa,$$

$$\frac{d^2\zeta_\kappa}{dt^2} + \frac{\mu^2}{r^3}\zeta_\kappa = \mu^2 Z_\kappa,$$

$$\kappa = 1, 2, 3, \cdots,$$

where

$$\Xi_\kappa = \vec{r}^\circ \cdot \vec{F}_\kappa,$$

$$H_\kappa = \vec{R}\times\vec{r}^\circ\cdot\vec{F}_\kappa,$$

$$Z_\kappa = \vec{R}\cdot\vec{F}_\kappa,$$

which are of the same form as Equations 7 through 9. Thus, our problem now is to integrate the variational equations of the form 7-9 with the right sides known.

INTEGRATION PROCEDURE

We shall make use of the substitution

$$\xi = ur + \frac{dr}{dt}\int(w - 2u)\,dt\,, \tag{15}$$

$$\eta = \frac{\mu\sqrt{p}}{r}\int(w - 2u)\,dt\,, \tag{16}$$

$$\zeta = \zeta\,, \tag{17}$$

which reduces Equations 7 through 9 to the form integrable by Hill's procedure. Substituting expressions 15 and 16 into 7, and making use of the relation

$$\frac{d^2 r}{dt^2} = \mu^2\left(\frac{p}{r^3} - \frac{1}{r^2}\right),$$

we obtain

$$\frac{d^2 u}{dt^2} + \frac{\mu^2}{r^3}(u - 2w) + \frac{1}{r}\frac{dr}{dt}\frac{dw}{dt} = \frac{\mu^2\,\Xi}{r}\,. \tag{18}$$

From Equations 15 and 16, we have

$$w = \frac{1}{\mu\sqrt{p}}\left(r\frac{d\eta}{dt} - \eta\frac{dr}{dt}\right) + \frac{2\xi}{r}\,. \tag{19}$$

Differentiating Equation 19 and taking 8 into consideration, we obtain

$$\frac{dw}{dt} = \frac{\mu r\,H}{\sqrt{p}} \tag{20}$$

and

$$w = K_3 + \int\frac{\mu r\,H}{\sqrt{p}}\,dt\,, \tag{21}$$

where the integral sign represents the integral obtained in a formal manner; K_3 is the constant of integration.

Taking Equations 20, 21, and the equation

$$\frac{dr}{dt} = \frac{\mu e \sin f}{\sqrt{p}}$$

into account, we obtain from Equation 18:

$$\frac{d^2}{dt^2}\left(u - 2K_3\right) + \frac{\mu^2}{r^3}\left(u - 2K_3\right) = \frac{\mu^2}{r}\left(\Xi - \frac{re \sin f}{p} H\right) + \frac{2}{r^3}\int \frac{\mu^3\, rH}{\sqrt{p}}\, dt \ .$$

The last equation can be integrated by using Hill's procedure, and we have

$$u = 2K_3 + K_1 \frac{r}{a} \cos f + K_2 \frac{r}{a} \sin f$$

$$+ \int \frac{\mu}{\sqrt{p}}\left(\frac{\Xi}{r} - \frac{e \sin f}{p} H\right)\bar{r}\, r \sin\left(\bar{f} - f\right) dt$$

$$+ \int \frac{\bar{r}}{r^2} \sin\left(\bar{f} - f\right) dt \int \frac{2\mu^2\, r\, H}{p}\, dt \ , \tag{22}$$

where K_1, K_2 are constants of integration and $\bar{r}$, $\bar{f}$ are considered as temporary constants; after the integration is completed, they are replaced by r and f.

The double integral in Equation 22 can be simplified through integration by parts, and we obtain

$$\int \frac{\bar{r}}{r^2} \sin\left(\bar{f} - f\right) dt \int \frac{2\mu^2\, rH}{p}\, dt = \int \frac{2\mu\, rH}{\sqrt{p}}\, dt - \int \frac{2\mu\, \bar{r} rH}{p\sqrt{p}}\left[\cos\left(\bar{f} - f\right) + e \cos \bar{f}\right] dt \ . \tag{23}$$

Taking this last relation into consideration, we deduce from Equation 22 after some easy transformations

$$u = 2K_3 + K_1 \frac{r}{a} \cos f + K_2 \frac{r}{a} \sin f + A \ , \tag{24}$$

in which

$$A = \int \left(M \vec{F} \cdot \vec{r}^{\,o} + N \vec{F} \cdot \vec{R} \times \vec{r}\right) dt \ ,$$

where

$$M = \frac{an}{\sqrt{1 - e^2}}\, \bar{r} \sin\left(\bar{f} - f\right) \ ,$$

$$N = \frac{an}{\left(1 - e^2\right)^{3/2}} \frac{\bar{r}}{a}\left[+ \frac{1}{2} e \cos \bar{f} - \frac{1}{2} e \cos\left(\bar{f} - 2f\right) - 2 \cos\left(\bar{f} - f\right) + 2\right] .$$

The expressions M and N remain the same for the perturbations of all orders. They are to be developed into a double Fourier series with respect to the mean anomaly l and with respect to the auxiliary mean anomaly $\bar{l}$, associated with the auxiliary true anomaly $\bar{f}$. After the integration is performed, $\bar{l}$ is replaced by l.

We have for the undisturbed ellipse

$$\int \frac{r}{a} \cos f \, dl = -\frac{3}{2} \text{ent} + \frac{1}{2} \sqrt{1-e^2} \frac{r}{a} \sin f + \frac{1}{2\sqrt{1-e^2}} \frac{r^2}{a^2} \sin f \quad , \tag{25}$$

$$\int \frac{r}{a} \sin f \, dl = -\sqrt{1-e^2} \frac{r^2}{a^2} \left(\cos f + \frac{1}{2} e \cos^2 f\right) \quad . \tag{26}$$

Putting

$$S = \int (w - 2u) \, dt$$

and taking Equations 21, 24, 25, and 26 into account, we deduce

$$S = -\frac{3K_3}{n} nt + \frac{K_1}{n} \left(3 \text{ ent} - \sqrt{1-e^2} \frac{r}{a} \sin f - \frac{1}{\sqrt{1-e^2}} \frac{r^2}{a^2} \sin f\right)$$
$$+ \sqrt{1-e^2} \frac{K_2}{n} \frac{r^2}{a^2} \left(2 \cos f + e \cos^2 f\right) + K_4 + B \; ;$$

and, after some easy transformations,

$$S = \frac{3}{n} \left(-K_3 + eK_1\right) nt - \frac{K_1}{n\sqrt{1-e^2}} \frac{r^2}{a^2} \left(2 \sin f + \frac{1}{2} e \sin 2f\right)$$
$$+ \frac{K_2 \sqrt{1-e^2}}{n} \frac{r^2}{a^2} \left(2 \cos f + \frac{1}{2} e + \frac{1}{2} e \cos 2f\right) + K_4 + B \quad , \tag{27}$$

where

$$B = \iint \frac{narH}{\sqrt{1-e^2}} dt^2 - \int 2A \, dt \quad . \tag{28}$$

The formulas for the computation of ξ, η, and ζ become

$$\xi = ru + \frac{nae \sin f}{\sqrt{1-e^2}} S \quad , \tag{29}$$

$$\eta = \frac{na^2\sqrt{1-e^2}}{r} S \quad , \tag{30}$$

$$\zeta = K_5 \frac{r}{a} \cos f + K_6 \frac{r}{a} \sin f + C \quad , \tag{31}$$

where C is the standard expression

$$C = \int \frac{na}{\sqrt{1-e^2}} \vec{R} \cdot \vec{F} \bar{r} \sin\left(\bar{f} - f\right) dt \quad . \tag{32}$$

The expressions for A, B, and C as given here are reducible to the form given in the author's previous article. The present form, however, facilitates the comparison with the perturbations already computed by Hill's method (Reference 5) if the substitution

$$dt = \frac{r^2}{a^2} \frac{df}{n\sqrt{1-e^2}}$$

is made.

DETERMINATION OF CONSTANTS OF INTEGRATION

We consider here the determination of constants of integration for the case when the elements are osculating at the epoch $t = 0$ and for the case when they are mean. If the elements are osculating, then we have

$$(\delta\vec{r})_0 = 0 \, , \qquad \left(\frac{d\delta\vec{r}}{dt}\right)_0 = 0 \, , \tag{33}$$

where the zero subscript designates the value of the expression at the epoch. From Equation 33 we deduce

$$u_0 = 0 \, , \qquad w_0 = 0 \, , \qquad \left(\frac{du}{dt}\right)_0 = 0 \, , \qquad S_0 = 0 \, . \tag{34}$$

Taking Equation 21 into account, we obtain

$$K_3 = -\left[\int \frac{naH}{\sqrt{1-e^2}} dt\right]_0 \quad . \tag{35}$$

Differentiating Equation 24 and considering the equations

$$\frac{d}{dl} \frac{r}{a} \cos f = -\frac{\sin f}{\sqrt{1-e^2}} \quad ,$$

$$\frac{d}{dl}\frac{r}{a}\sin f = +\frac{\cos f + e}{\sqrt{1-e^2}} ,$$

we deduce from Equation 34:

$$+ K_1 \frac{r_0}{a}\cos f_0 + K_2 \frac{r_0}{a}\sin f_0 = -A_0 - 2K_3 ,$$

$$- K_1 \frac{\sin f_0}{\sqrt{1-e^2}} + K_2 \frac{\cos f_0 + e}{\sqrt{1-e^2}} = -A_0' ,$$

where we put

$$A_0' = \left(\frac{dA}{dl}\right)_0 .$$

From these last equations we obtain

$$K_1 = -\frac{\cos f_0 + e}{1 - e^2}\left(A_0 + 2K_3\right) + \frac{A_0'}{\sqrt{1-e^2}}\frac{r_0}{a}\sin f_0 ,$$

$$K_2 = -\frac{\sin f_0}{1 - e^2}\left(A_0 + 2K_3\right) - \frac{A_0'}{\sqrt{1-e^2}}\frac{r_0}{a}\cos f_0 .$$

In a similar way we deduce

$$K_5 = -\frac{\cos f_0 + e}{1 - e^2} C_0 + \frac{C_0'}{\sqrt{1-e^2}}\frac{r_0}{a}\sin f_0 ,$$

$$K_6 = -\frac{\sin f_0}{1 - e^2} C_0 - \frac{C_0'}{\sqrt{1-e^2}}\frac{r_0}{a}\cos f_0 ;$$

and, putting $t = 0$ in Equation 27, we deduce the following value for K_4:

$$K_4 = +\frac{K_1}{n\sqrt{1-e^2}}\frac{r_0^2}{a^2}\left(2\sin f_0 + \frac{1}{2} e_0 \sin 2f_0\right)$$

$$-\frac{K_2\sqrt{1-e^2}}{n}\frac{r_0^2}{a^2}\left(2\cos f_0 + \frac{1}{2} e + \frac{1}{2} e\cos 2f_0\right) - B_0 .$$

The mean elements can be defined in several ways. We accept here the following definition. The elements are mean if:

1. The perturbations of the true longitude λ with respect to the orbit plane defined by these elements do not contain the terms of the form

$$K_0,\ Kt,\ K^{(c)}\cos l,\ \text{and}\ K^{(s)}\sin l. \tag{36}$$

2. The expression for the "third coordinate" ζ does not contain the terms of the form

$$K^{(c)} \cos l, \qquad K^{(s)} \sin l \quad .$$

We have

$$\left(\frac{r}{a}\right)^n \cos mf = \frac{1}{2} C_0^{n,m} + C_1^{n,m} \cos l + C_2^{n,m} \cos 2l + \cdots \quad ,$$

$$\left(\frac{r}{a}\right)^n \sin mf = S_1^{n,m} \sin l + S_2^{n,m} \sin 2l + \cdots \quad ,$$

where

$$C_i^{n,m} = X_i^{n,m} + X_{-i}^{n,m} \quad ,$$

$$S_i^{n,m} = X_i^{n,m} - X_{-i}^{n,m} \quad ,$$

and $X_i^{n,m}$ are Hansen's coefficients.

Let us start with the determination of constants of integration in the perturbations of the first order. Perturbations of the first order in the true longitude are given by the expression η/r, where $\eta = \eta_1$, and consequently the terms of the form 36 must be absent in the expression

$$\begin{aligned} \frac{\eta}{nr\sqrt{1-e^2}} = {} & \frac{3}{n}\left(\frac{a}{r}\right)^2 \left(-K_3 + eK_1\right) nt \\ & - \frac{K_1}{n\sqrt{1-e^2}} \left(2 \sin f + \frac{1}{2} e \sin 2f\right) \\ & + \frac{K_2\sqrt{1-e^2}}{n} \left(2 \cos f + \frac{1}{2} e + \frac{1}{2} e \cos 2f\right) \\ & + \frac{a^2}{r^2} K_4 + \frac{a^2}{r^2} B \quad . \end{aligned} \tag{37}$$

We have to separate the terms of the form 36 in the development of $\left(a^2/r^2\right)B$:

$$\frac{a^2}{r^2} B = \alpha_0 + \beta_0 nt + \alpha_1 \cos l + \beta_1 \sin l + \cdots \quad . \tag{37'}$$

The condition for absence of the constant term in Equation 37 leads to

$$+ \frac{K_2\sqrt{1-e^2}}{n} \left(+C_0^{0,1} + \frac{1}{2} e + \frac{1}{4} e C_0^{0,2}\right) + \frac{1}{2} C_0^{-2,0} K_4 + \alpha_0 = 0 \quad . \tag{38}$$

In a similar way we deduce

$$-\frac{K_1}{n\sqrt{1-e^2}}\left(2S_1^{0,1}+\frac{1}{2}\,e\,S_1^{0,2}\right)+\beta_1 = 0 \quad , \tag{39}$$

$$+\frac{K_2\sqrt{1-e^2}}{n}\left(2C_1^{0,1}+\frac{1}{2}\,e\,C_1^{0,2}\right)+\alpha_1 = 0 \quad , \tag{40}$$

$$+\frac{3}{2n}\,C_0^{-2,0}\left(-K_3+e\,K_1\right)+\beta_0 = 0 \quad . \tag{41}$$

Separating in C the terms with the argument l,

$$C = c_1 \cos l + s_1 \sin l + \cdots \quad ,$$

we obtain, taking Equation 31 into account,

$$K_5\,C_1^{1,1} + c_1 = 0 \quad , \tag{42}$$

$$K_6\,S_1^{1,1} + s_1 = 0 \quad . \tag{43}$$

In the planetary case the solutions of Equations 38 through 43 can be found without any difficulty, because the coefficients of only one unknown in each equation are not small. The coefficients $C_i^{n,m}$ can be computed either by using the classical analytical expressions, by Cayley's tables (Reference 6), or by means of harmonic analysis. The latter procedure is preferable if the eccentricity is not very small.

Determination of constants of integration in higher order perturbations requires some additional considerations. Let r_1, r_2, $\cdots$ be the perturbations in the radius vector r and λ_1, λ_2, $\cdots$ be the perturbations in the true longitude λ of the first, second, etc. orders. From

$$\left(\xi_1+\xi_2+\cdots\right)\vec{r}^{\,\circ}+\left(\eta_1+\eta_2+\cdots\right)\vec{R}\times\vec{r}^{\,\circ} = \left(r_1\frac{\partial\vec{r}}{\partial r}+\lambda_1\frac{\partial\vec{r}}{\partial\lambda}\right)$$

$$+\left[\left(r_2\frac{\partial\vec{r}}{\partial r}+\lambda_2\frac{\partial\vec{r}}{\partial\lambda}\right)+\frac{1}{2}\left(r_1^2\frac{\partial^2\vec{r}}{\partial r^2}+2r_1\lambda_1\frac{\partial^2\vec{r}}{\partial r\partial\lambda}+\lambda_1^2\frac{\partial^2\vec{r}}{\partial\lambda^2}\right)\right]+\cdots$$

and substituting

$$\frac{\partial\vec{r}}{\partial r} = \vec{r}^{\,\circ} \,,\quad \frac{\partial\vec{r}}{\partial\lambda} = \vec{R}\times\vec{r} \,,\quad \frac{\partial^2\vec{r}}{\partial r^2} = 0 \,,\quad \frac{\partial^2\vec{r}}{\partial r\partial\lambda} = \vec{R}\times\vec{r}^{\,\circ} \,,\quad \frac{\partial^2\vec{r}}{\partial\lambda^2} = -\vec{r} \,,$$

we obtain

$$\xi_1 = r_1 \, , \qquad \eta_1 = r\lambda_1 \, ,$$

$$\xi_2 = r_2 - \frac{1}{2}\lambda_1^2 r \, , \qquad \eta_2 = r\lambda_2 + r_1\lambda_1 \quad .$$

Consequently,

$$\lambda_2 = \frac{\eta_2 - \xi_1 \eta_1}{r}$$

or, taking Equation 30 into account,

$$\lambda_2 = n\sqrt{1-e^2}\,\frac{a^2}{r^2}\left(S_2 - \xi_1 S_1\right) \quad ,$$

where S_1 corresponds to the first and S_2 to the second order perturbations. As we see, in the determination of the constants of integration of the second order in the case of mean elements a correction term $-\xi_1 S_1$ must be added to B_2. For the perturbation of the third order a similar correction term will depend on ξ_1, S_1, ξ_2, S_2.

CONCLUSION

A revival of the general interest in planetary theories can be observed in our time. Several scientific institutions are dedicating their time and efforts to the astronomical solution of the planetary problem. The results by Brouwer (Reference 7), Gontkovskaya (Reference 8), and Danby (Reference 9) must especially be mentioned. A considerable amount of work on the theoretical exposition as well as on programming also has been done at Goddard Space Flight Center. In the present article we suggest a new scheme which is convenient for computing the perturbations of the first as well as higher orders.

The determination of constants of integration in the case of both the osculating and the mean elements is a straightforward process in the proposed scheme. An important feature of the scheme is that the squares of small divisors, as caused by the commensurability of mean motions, are introduced by integration of only one expression, namely $w - 2u$. The short and the long period terms containing the squares of small divisors constitute a significant part in the perturbations ξ, η, ζ. The existence of such a direct way of separating these terms from the remaining perturbations constitutes a significant part of a planetary theory. The development here is kept in the form which facilitates comparison with the results obtained on the basis of the classical form of Hill's theory, if necessary.

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Appendix A

Notations

a	undisturbed semimajor axis of planet m
E	base of natural logarithms
e	undisturbed eccentricity of planet m
f	undisturbed true anomaly of planet m
I	idemfactor
k	Gaussian constant
l	undisturbed mean anomaly of planet m
m	mass of disturbed planet; the mass of the sun is taken as unity
m'	mass of disturbing planet
n	$\frac{\mu}{a^{3/2}}$, undisturbed mean motion of planet m
p	$a\left(1 - e^2\right)$
$\vec{R}$	unit vector normal to undisturbed orbit plane of planet m
r	$\lvert\vec{r}\rvert$
$\vec{r}$	undisturbed position vector of planet m
$\vec{r}\,'$	undisturbed position vector of planet m'
$\vec{r}^{\,\circ}$	unit vector in direction of $\vec{r}$
$\vec{r}_K$	perturbations of k^{th} order in position vector of planet m
$\vec{r}_K'$	perturbations of k^{th} order in position vector of planet m'
$\vec{r} + \delta\vec{r}$	disturbed position vector of planet m
$\vec{r}\,' + \delta\vec{r}\,'$	disturbed position vector of planet m'

$\vec{v}$	undisturbed velocity of planet m
∇	del-operator with respect to $\vec{r}$
∇'	del-operator with respect to $\vec{r}'$
$\delta\vec{r}$	perturbations in position vector of planet m
$\delta\vec{r}'$	perturbations in position vector of planet m'
μ^2	$k^2(1+m)$
$\vec{\rho}$	$\vec{r}' - \vec{r}$
$\vec{\rho}_K$	$\vec{r}'_K - \vec{r}_K$
$\vec{\psi}$	angular velocity of rotation of frame $(\vec{r},\ \vec{R}\times\vec{r},\ \vec{R})$

$\Omega(\vec{r},\ \vec{r}') = \frac{m'}{1+m}\left(\frac{1}{\rho} - \frac{\vec{r}\cdot\vec{r}'}{r'^3}\right)$ — main part of disturbing function

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